An action of the cactus group

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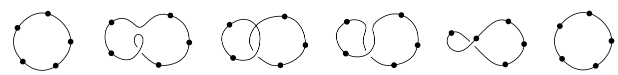
Let $\overline{M}_{0,n}(\mathbb{R})$ denote the Deligne-Mumford compactification of the moduli space of real curves of genus zero with n marked points. Its points are the isomorphism classes of stable real curves of genus zero, that is, curves obtained by glueing $\mathbb{RP}^1$'s in a tree-like way, and such that each irreducible component has at least 3 special points. Let $[\overline{M}_{0,n+1}(\mathbb{R})/S_n]$ denote the quotient orbifold of $\overline{M}_{0,n+1}(\mathbb{R})$ by the action permuting the first n marked points. In [3], J. Kamnitzer and the author showed that the *cactus group* $J_n := \pi_1([\overline{M}_{0,n+1}(\mathbb{R})/S_n])$ acts on tensor powers of Kashiwara crystals in a way similar to how the braid group acts on tensor powers of quantum group representations.

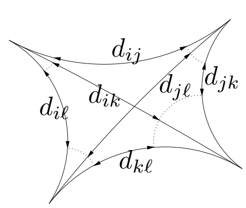
The *big cactus group* J'_n is the fundamental group of $[\overline{M}_{0,n}(\mathbb{R})/S_n]$. It fits into a short exact sequence $0 \to \pi_1(\overline{M}_{0,n}(\mathbb{R})) \to J'_n \to S_n \to 0$, and its elements can be represented by movies, such as the following one:



Let $\mathcal{F}\ell_m := \left(\begin{smallmatrix}1 & & * \\ & 1 & \\ & & 1\end{smallmatrix}\right)\backslash SL_m$ be the variety of flags $0 \subset V_1 \subset \cdots \subset V_{m-1} \subset \mathbb{R}^m$, equipped with volume forms $\omega_i \in \Lambda^i V_i$. The goal of this note is to construct an action of J'_n on the totally positive part $\mathcal{A}(n)_{>0}$ of the variety $\mathcal{A}(n) := (\mathcal{F}\ell_m)^n/SL_m$. The space $\mathcal{A}(n)_{>0}$ is a certain connected component of the locus $\mathcal{A}(n)_{reg} \subset \mathcal{A}(n)$, where the flags are in generic position. One gets similar actions on $\big((N\backslash G)^n/G\big)_{>0}$ for other reductive groups G.

The space $\mathcal{A}(n)_{>0}$ was introduced by Fock and Goncharov [1]. For $m = 2$, it agrees with the Teichmüller space of decorated ideal n-gons, that is, the space of isometry classes of hyperbolic n-gons with geodesic sides, vertices at infinity, and horocycles around each vertex. It is also an example of a cluster variety, i.e. it comes with special sets of coordinate systems, whose transition functions are given by *cluster exchange relations* [2]. For $m = 2$, the coordinates are due to Penner [4]. To each pair i, j of vertices of the n-gon, he associates the quantity $\Delta_{ij} := \exp(\frac{1}{2}d_{ij})$, where d_{ij} denotes the hyperbolic length between the intersection points of the horocycles around i and j, and the geodesic from i to j. These coordinates are then subject to the following exchange relations [4]:

(1)



$$\Delta_{j\ell} = \frac{\Delta_{ij}\Delta_{k\ell} + \Delta_{jk}\Delta_{i\ell}}{\Delta_{ik}}.$$

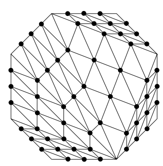
For general m, the coordinates on $\mathcal{A}(n)$ are indexed by tuples $(i_1, \ldots, i_n) \in \mathbb{N}^n$ whose sum equals m, and such that at least two entries are non-zero. The coordinate $\Delta_{i_1 \ldots i_n}$ then assigns to $((V^1_\bullet, \omega^1_\bullet), \ldots, (V^n_\bullet, \omega^n_\bullet)) \in (\mathcal{F}\ell_m)^n$ the ratio of

$\omega^1_{i_1} \wedge \cdots \wedge \omega^n_{i_n}$ with the standard volume form on $\mathbb{R}^m$. These coordinates satisfy

$$\Delta_{\ldots i \ldots j \ldots k \ldots \ell \ldots} = \big(\Delta_{\ldots i+1 \ldots j \ldots k \ldots \ell-1 \ldots} \cdot \Delta_{\ldots i \ldots j-1 \ldots k+1 \ldots \ell \ldots} + \\ \Delta_{\ldots i \ldots j \ldots k+1 \ldots \ell-1 \ldots} \cdot \Delta_{\ldots i+1 \ldots j-1 \ldots k \ldots \ell \ldots}\big) \big/ \Delta_{\ldots i+1 \ldots j-1 \ldots k+1 \ldots \ell-1 \ldots}\ ,$$

which generalizes (1). Let $\mathcal{A}(n)_{>0}$ be the locus where all the Δ's are > 0. It is a space isomorphic to $\mathbb{R}_{>0}^{(n-2)\cdot\binom{m+1}{2}+(m+1)-n}$, and each triangulation of the n-gon provides such an isomorphism [1]. More precisely, the isomorphism corresponding to a triangulation is given by the coordinates $\Delta_{0\ldots 0i0\ldots 0j0\ldots 0k0\ldots 0}$, where i, j, k are located at the vertices of the triangles. For example, for $n = 8$, $m = 4$, and the triangulation  of the 8-gon, the corresponding coordinates on $\mathcal{A}(n)_{>0}$ are in natural bijection with the bullets in the following figure:

(2)



We now explain a general machine for producing actions of J'_n on various spaces. Suppose that we are given two manifolds $X_\triangle$ and X_I, equipped with maps

$$(3) \qquad r \circlearrowright X_\triangle \underset{d_3=d_0}{\overset{d_1}{\underset{\xrightarrow{d_2}}{\rightarrow}}} X_I \circlearrowleft \iota$$

subject to the relations $r^3 = 1$, $\iota^2 = 1$, and $d_i \circ r = r \circ d_{i-1}$. Such data can then be reinterpreted as a contravariant functor $X_\bullet : \mathcal{C} \to \{\text{manifolds}\}$ from the category $\mathcal{C} := \{ \circlearrowright \triangle \leftleftarrows I \circlearrowleft \}$, whose two objects are the oriented triangle "$\triangle$" and the unoriented interval "I", and whose morphisms are the obvious embeddings and automorphisms. Let $\widehat{\mathcal{C}}$ be the category whose objects are the 2-dimensional finite simplicial complexes with oriented 2-faces and connected links, and whose morphisms are the embeddings. There is an obvious inclusion $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$, and every object of $\widehat{\mathcal{C}}$ can be written essentially uniquely as the colimit of a diagram in $\mathcal{C}$. Assuming $d_1 \times d_2 \times d_3 : X_\triangle \to X_I^3$ is a submersion, then there is a unique extension of $X_\bullet$ to $\widehat{\mathcal{C}}$ sending colimits to limits. For example, using that extension, we get $X_{\boxslash} \cong X_\triangle \times_{X_I} X_\triangle$.

Theorem 1. *Let $X_\bullet$ be a functor as above, and denote by the same letter its canonical extension to $\widehat{\mathcal{C}}$. Suppose that we are given isomorphisms*

$$\tau : X_{\boxslash} \to X_{\boxbslash} \qquad \textit{and} \qquad \theta : X_\triangle \to X_\triangle$$

making the following diagrams commute:

1)
$$\begin{tikzcd} X_{\boxslash} \arrow[rr, "\tau"] \arrow[dr, "\prod^4 d'_i"'] & & X_{\boxbslash} \arrow[dl, "\prod^4 d''_i"] \\ & X_I^4 & \end{tikzcd}$$
where $d'_i : X_{\boxslash} \to X_I$, $d''_i : X_{\boxbslash} \to X_I$, $i = 1..4$, are induced by the four face inclusions $I \hookrightarrow \square$.

2) $$\begin{tikzcd} X_{\boxslash} \arrow[r, "\tau"] \arrow[d, "1/2"'] & X_{\boxbslash} \arrow[d, "1/2"] \\ X_{\boxslash} \arrow[r, "\tau"] & X_{\boxbslash} \end{tikzcd}$$ *where* $1/2 : X_{\boxslash} \to X_{\boxslash}$ *and* $1/2 : X_{\boxbslash} \to X_{\boxbslash}$ *are induced by half turn rotation of the square.*

3) $$\begin{tikzcd} X_{\boxslash} \arrow[r, "\tau"] \arrow[d, "1/4"'] & X_{\boxbslash} \arrow[d, "1/4"] \\ X_{\boxbslash} & X_{\boxslash} \arrow[l, "\tau"'] \end{tikzcd}$$ *where* $1/4 : X_{\boxslash} \to X_{\boxbslash}$ *and* $1/4 : X_{\boxbslash} \to X_{\boxslash}$ *are induced by rotation by a quarter turn.*

4) $$\begin{tikzcd} & X_{\pentagon} \arrow[r, "\tau\times 1"] & X_{\pentagon} \arrow[dr, "1\times\tau"] & \\ X_{\pentagon} \arrow[ur, "1\times\tau"] \arrow[r, "\tau\times 1"] & X_{\pentagon} \arrow[rr, "1\times\tau"] & & X_{\pentagon} \end{tikzcd}$$ *note that "*$1 \times \tau$*" and "*$\tau \times 1$*" only become well defined once we have axioms 1) and 2).*

5) $d_i \circ \theta = d_{4-i}$

6) $\theta \circ r = r^{-1} \circ \theta$

7) $\theta^2 = 1$,

8) $$\begin{tikzcd} X_{\boxslash} \arrow[r, "\tau"] \arrow[d, "\theta\times\theta"'] & X_{\boxbslash} \arrow[d, "\theta\times\theta"] \\ X_{\boxslash} \arrow[r, "\tau\circ 1/2"] & X_{\boxbslash} \end{tikzcd}$$

then there is a natural action of J_n' *on the manifold that* $X_\bullet$ *associates to a triangulated* n*-gon. (For example, one gets an action of* J_8' *on* $X_{\octagon}$*).*

We now use the above theorem to equip $\mathcal{A}(n)_{>0}$ with a J_n' action. Indeed, the manifolds $X_\triangle := \mathcal{A}(3)_{>0}$ and $X_I := \mathcal{A}(2)_{>0}$ fit into a diagram (3), and so provide a functor $\widehat{\mathcal{C}} \to \{\text{manifolds}\}$. The space associated to a triangulated n-gon is $\mathcal{A}(n)_{>0}$, as can be seen from the parameterization (2). We let τ be the composite

$$\tau : X_{\boxslash} \xrightarrow{\sim} \mathcal{A}(4)_{>0} \xrightarrow{\sim} X_{\boxbslash},$$

and θ be the map sending $(F_1, F_2, F_3) \in (\mathcal{F}\ell_m)^3$ to $(F_3^\perp, F_2^\perp, F_1^\perp)$, where the orthogonal of a flag F is given by $(V_1, \ldots, V_{m-1})^\perp := (V_{m-1}^\perp, \ldots, V_1^\perp)$, along with $\pm$ the obvious volume forms. The axioms *1)–8)* are then easy to check.

Both τ and θ are composites of cluster exchange relations. But the action of J_n' on $\mathcal{A}(n)_{>0}$ is not cluster (it doesn't satisfy the Laurent phenomenon; it doesn't preserve the canonical presymplectic form). The reason is that θ is actually the composite of a cluster map with an automorphism that negates the cluster matrix. In particular, it negates the presymplectic form.

References

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[3] A. Henriques, J. Kamnitzer, *Crystals and coboundary categories*, Duke Math. J. **132** no. 2 (2006), 191–216.

[4] R. C. Penner, *The decorated Teichmüller space of punctured surfaces* Comm. Math. Phys. **113** no.2 (1987), 299–339.